

// EXERCISE 7.8 //

1 (i) compute $\sqrt{(4.5)} = \sqrt{9} = \frac{7}{9} \cdot \frac{5}{8} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{1} = \frac{105}{16} \sqrt{1}$

$$= \frac{105}{16} \times 1.77245 \dots$$

$$= 11.6317 \text{ Ans}$$

(ii) $\sqrt{\frac{3}{2}} =$

using $\sqrt{n} = \frac{\sqrt{(n+1)}}{n}$

$$\sqrt{\frac{3}{2}} = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{3}} = \frac{2}{3} \sqrt{\frac{1}{2}} \dots \textcircled{1}$$

from (i) and (ii)

$$\sqrt{\frac{3}{2}} = \left(\frac{2}{3}\right) \left(\frac{2}{1}\right) \sqrt{1}$$

$$= \frac{4}{3} \sqrt{1} \text{ Ans}$$

Now $\sqrt{\frac{1}{2}} = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}} = -\frac{2}{1} \sqrt{\frac{1}{2}} = -2\sqrt{1} \dots \textcircled{II}$

$$(iii) \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\text{using } \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} = \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{\Gamma(1)} = \pi$$

$$(iv) \beta(2.5, 1.5) = \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2} + \frac{3}{2}\right)} = \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\pi}}{\Gamma(4)} = \frac{\frac{3}{8} \pi}{12} = \frac{\pi}{16}$$

$$= \frac{3}{8} \pi = \frac{3}{8} \cdot \frac{1}{6} \pi = \frac{1}{16} \pi = \frac{3.141592}{16}$$

$$= 0.196349 \text{ (Approx.)}$$

$$(v) \beta\left(\frac{1}{2}, \frac{7}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{7}{2}\right)} = \frac{\sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{\Gamma(4)} = \frac{\frac{15}{8} \pi}{12} = \frac{15}{96} \pi = \frac{5}{32} \pi \text{ Ans}$$

Q2. Express the following Integrals in terms of gamma function

$$(i) \int_0^{\infty} e^{-x^2} dx \quad \text{or show that } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\text{put } x^2 = z \text{ so that } 2xdx = dz \text{ or } xdx = \frac{dz}{2}$$

$$\therefore \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} \frac{e^{-z} dz}{2\sqrt{z}} = \frac{1}{2} \int_0^{\infty} e^{-z} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\text{put } x = -z \text{ so that } dx = -dz \quad \text{as } x \rightarrow -\infty, z \rightarrow \infty$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{\infty}^0 e^{-z} (-dz) = \int_0^{\infty} e^{-z} dz = \frac{\sqrt{\pi}}{2}$$

show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\int_0^{\infty} x^6 e^{-3x} dx \quad \text{put } 3x = z, \quad dx = \frac{dz}{3}$$

$$\int_0^{\infty} \left(\frac{z}{3}\right)^6 e^{-z} \frac{dz}{3} = \frac{1}{3^7} \int_0^{\infty} e^{-z} z^{7-1} dz = \frac{1}{3^7} \Gamma(7) = \frac{16}{2187}$$

$$(iii) \int_0^{\infty} x e^{-x^3} dx$$

$$\text{put } x^3 = z, \quad x = z^{\frac{1}{3}}$$

$$dx = \frac{1}{3} z^{-\frac{2}{3}} dz$$

$$\therefore \int_0^{\infty} (z^{\frac{1}{3}}) \cdot e^{-z} \cdot \frac{1}{3} z^{-\frac{2}{3}} dz = \int_0^{\infty} \frac{e^{-z} z^{\frac{1}{3}-1}}{3} dz$$

$$= \frac{1}{3} \int_0^{\infty} e^{-z} z^{\frac{1}{3}-1} dz = \frac{1}{3} \Gamma\left(\frac{1}{3}\right) = \frac{\sqrt{\pi}}{3}$$

(iv) $\int_0^{\infty} \frac{dx}{x^{p+1} (x-1)}$ Not match with the Answer

$p+2-1 \beta(p,q)$

So we have Accurate Statement

$\therefore \int_{-1}^1 \frac{x^{p-1}}{(1+x)(1-x)} dx$

put $1+x=z$, $dx=dz$

$x=z-1$
 $1-x=1-z+1=2-z$

$\therefore \int_0^1 \frac{(z-1)^{p-1}}{(z-z+1)(2-z)} dz = \int_0^1 \frac{(z-1)^{p-1}}{z(2-z)} dz$

$= \int_0^1 \frac{(z-1)^{p-1}}{z} dz = \int_0^1 \frac{(z-1)^{p-1}}{z} dz$

(33 (i))

Show that $\int_0^{\infty} \frac{x^p}{e^x} dx = \frac{\Gamma(p+1)}{\Gamma(p+1)}$

use $a^x = e^{x \log a}$

$\therefore \int_0^{\infty} \frac{x^p}{e^{x \log a}} dx$

put $x \log a = z$, so that $dx = \frac{dz}{\log a}$

$\therefore \int_0^{\infty} \left(\frac{z}{\log a} \right)^p \cdot \frac{1}{z} \cdot \frac{dz}{\log a}$

$= \frac{1}{(\log a)^{p+1}} \int_0^{\infty} \frac{z^p}{e^z} dz = \frac{\Gamma(p+1)}{(\log a)^{p+1}}$

(ii) Show that $\int_0^{\pi} \sqrt{\cos \theta} d\theta = \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{3}{4}}$

$\therefore \int_0^{\pi} \cos^{\frac{1}{2}} \theta \sin^{\frac{1}{2}} \theta d\theta$

using $\int_0^{\pi} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma(p+1) \Gamma(q+1)}{2 \Gamma(p+q+2)}$

$\therefore \int_0^{\pi} \cos^{\frac{1}{2}} \theta \sin^{\frac{1}{2}} \theta d\theta = \frac{\Gamma(\frac{1}{2}+1) \Gamma(\frac{1}{2}+1)}{2 \Gamma(1+1)} = \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{3}{4}}$

(iii) Show that $\int_0^{\pi} \sqrt{\tan \theta} + \sqrt{\sec \theta} d\theta = \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{3}{4}} + \frac{\sqrt{\pi}}{2}$

$\therefore \int_0^{\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{\frac{1}{2}} + \left(\frac{1}{\cos \theta} \right)^{\frac{1}{2}} d\theta$

$= \int_0^{\pi} (\sin \theta)^{-\frac{1}{2}} (\cos \theta)^{-\frac{1}{2}} d\theta + \int_0^{\pi} (\cos \theta)^{-\frac{1}{2}} d\theta$

$= \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{1}{4}} + \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{3}{4}} = \frac{1}{2} \left\{ \sqrt{\frac{1}{4}} + \sqrt{\frac{3}{4}} \right\} = \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{3}{4}}$

(iv) Show that $\int_0^{\pi} (\sin \theta)^{\frac{1}{2}} d\theta \times \int_0^{\pi} \frac{d\theta}{\sin \theta} = \pi$

$\therefore \int_0^{\pi} (\sin \theta)^{\frac{1}{2}} \cos \theta d\theta \times \int_0^{\pi} (\sin \theta)^{-\frac{1}{2}} \cos \theta d\theta$

$= \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{1}{4}} \times \frac{1}{2} \sqrt{\frac{1}{4}} \sqrt{\frac{1}{4}} = \frac{1}{4} \sqrt{\frac{1}{4}} \sqrt{\frac{1}{4}} = \pi$

(v)

$$\text{Show that } \int_0^{\infty} e^{-x^4} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$$

(vi)

$$\text{Show that } \int_0^{\infty} x e^{-x^8} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$$

Sol

$$\text{Put } x^8 = z \Rightarrow x = z^{\frac{1}{8}} \\ \text{i.e. } dx = \frac{1}{8} z^{-\frac{7}{8}} dz$$

$$\int_0^{\infty} z^{\frac{1}{8}} \cdot e^{-z} \cdot \frac{1}{8} z^{-\frac{7}{8}} dz = \frac{1}{8} \int_0^{\infty} z^{-\frac{6}{8}} e^{-z} dz = \frac{1}{8} \int_0^{\infty} z^{-\frac{3}{4}} e^{-z} dz = \frac{1}{8} \Gamma\left(\frac{1}{4}\right) \quad \text{--- (i)}$$

$$\text{Now put } x^4 = t \text{ i.e. } x = t^{\frac{1}{4}} \\ dx = \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$\therefore \int_0^{\infty} t^{\frac{1}{4}} \cdot e^{-t} \cdot \frac{1}{4} t^{-\frac{3}{4}} dt = \frac{1}{4} \int_0^{\infty} t^{-\frac{1}{4}} e^{-t} dt = \frac{1}{4} \int_0^{\infty} t^{-\frac{1}{4}} e^{-t} dt = \frac{1}{4} \Gamma\left(\frac{3}{4}\right) \quad \text{--- (ii)}$$

$$\therefore \int_0^{\infty} x e^{-x^8} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{1}{8} \left[\frac{1}{4} \cdot \frac{1}{4} \right] \left[\frac{3}{4} \right] = \frac{1}{8} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \\ = \frac{1}{8} \times \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{1}{8} \times \frac{9}{16} = \frac{9}{128}$$

$$= \frac{\pi}{16\sqrt{2}}$$

$$\therefore \int_0^{\infty} x e^{-x^8} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$$

(vi)

$$\text{Show that } \int_0^{\infty} \sqrt{x} e^{-x^2} dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$$

$$\text{Sol. Put } x^2 = z, \quad x = z^{\frac{1}{2}} \\ dx = \frac{1}{2} z^{-\frac{1}{2}} dz$$

$$\therefore \int_0^{\infty} z^{\frac{1}{4}} \cdot e^{-z} \cdot \frac{1}{2} z^{-\frac{1}{2}} dz = \frac{1}{2} \int_0^{\infty} z^{-\frac{1}{4}} e^{-z} dz = \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \quad \text{--- (i)}$$

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Now

$$\text{we have } x^2 = z, \quad x = z^{\frac{1}{2}} \\ dx = \frac{1}{2} z^{-\frac{1}{2}} dz$$

$$\therefore \int_0^{\infty} \frac{e^{-z}}{z^{\frac{1}{2}}} \cdot \frac{1}{2} z^{-\frac{1}{2}} dz = \frac{1}{2} \int_0^{\infty} z^{-1} e^{-z} dz = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad \text{--- (ii)}$$

(iii)

$$\therefore \int_0^{\infty} \sqrt{x} \cdot e^{-x^2} dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{4} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} (\sqrt{2}\pi) = \frac{\pi}{2\sqrt{2}}$$

$$\therefore \int_0^{\infty} \sqrt{x} \cdot e^{-x^2} dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$$

(iii)

$$\text{Show that } \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy \times \int_0^{\infty} y^2 e^{-y^4} dy = \frac{\pi}{4\sqrt{2}}$$

$$\text{Sol. Put } y^2 = z, \quad y = z^{\frac{1}{2}}, \quad dy = \frac{1}{2} z^{-\frac{1}{2}} dz$$

$$\therefore \int_0^{\infty} \frac{e^{-z}}{z^{\frac{1}{4}}} \cdot \frac{1}{2} z^{-\frac{1}{2}} dz = \frac{1}{2} \int_0^{\infty} z^{-\frac{3}{4}} e^{-z} dz = \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \quad \text{--- (i)}$$

$$\text{Again put } y^4 = t, \quad y = t^{\frac{1}{4}}, \quad y^2 = t^{\frac{1}{2}} \\ dy = \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$\therefore \int_0^{\infty} t^{\frac{1}{2}} \cdot e^{-t} \cdot \frac{1}{4} t^{-\frac{3}{4}} dt = \frac{1}{4} \int_0^{\infty} t^{-\frac{1}{4}} e^{-t} dt = \frac{1}{4} \Gamma\left(\frac{3}{4}\right) \quad \text{--- (ii)}$$

$$\therefore \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy \times \int_0^{\infty} y^2 e^{-y^4} dy = \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4} \Gamma\left(\frac{3}{4}\right) = \frac{1}{8} (\sqrt{2}\pi) \\ = \frac{\pi}{4\sqrt{2}} \quad \text{Ans}$$

Show that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

Sol: Put $x^2 = \sin \theta$

$x = \sqrt{\sin \theta}$

$dx = \frac{1}{2\sqrt{\sin \theta}} \cdot \cos \theta d\theta$

$\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx = \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{2\sqrt{\sin \theta}} d\theta$

$= \frac{1}{2} \int_0^{\pi/2} \sin^{\frac{1}{2}} \theta \cos^0 \theta d\theta = \frac{1}{2} \cdot \left[\frac{\sin^{\frac{1}{2}+1} \theta}{\frac{1}{2}+1} \right]_0^{\pi/2}$

$= \frac{1}{2} \left[\frac{\sin^{\frac{3}{2}} \theta}{\frac{3}{2}} \right]_0^{\pi/2}$

$= \frac{1}{4} \left[\frac{\sqrt{3}}{\sqrt{5}} \right]_0^{\pi/2} = \frac{\sqrt{\pi}}{4} \cdot \frac{\sqrt{3}}{\sqrt{5}} = \frac{\pi}{4} \cdot \frac{\sqrt{3}}{\sqrt{5}}$

Now put $x^2 = \tan \phi$

$x = \sqrt{\tan \phi}$ so that $dx = \frac{\sec^2 \phi}{2\sqrt{\tan \phi}} d\phi$

$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2 \phi}} \cdot \frac{\sec^2 \phi}{2\sqrt{\tan \phi}} d\phi$

$= \frac{1}{2} \int_0^{\pi/4} \frac{d\phi}{\sqrt{2} \sin \phi \cos \phi} = \frac{1}{2\sqrt{2}} \int_0^{\pi/4} \frac{d\phi}{\sin \phi \cos \phi}$

$= \frac{\sqrt{2}}{2} \int_0^{\pi/4} \frac{d\phi}{\sin \phi \cos \phi} = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\phi}{\sin 2\phi}$

put $2\phi = t$

$d\phi = \frac{1}{2} dt$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sqrt{\sin t}} \cdot \frac{1}{2} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-\frac{1}{2}} t \cdot \cos^0 t dt$

$= \frac{1}{2\sqrt{2}} \cdot \left[\frac{\sin^{\frac{1}{2}+1} t}{\frac{1}{2}+1} \right]_0^{\pi/2} = \frac{1}{4\sqrt{2}} \left[\frac{\sqrt{2}}{\sqrt{2}} \right]_0^{\pi/2}$

$= \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{5}} = \frac{\pi}{4\sqrt{2}}$

$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4} \cdot \frac{\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{\pi}}{4\sqrt{2}} = \frac{\pi}{2} \cdot \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{\pi}}{4\sqrt{2}}$

$= \frac{\pi}{4\sqrt{2}} = R.H.S$

(ix) show that $\int_0^{\pi/2} \sin^p x \times \int_0^{\pi/2} \sin^q x dx = \frac{\pi}{2} (p+1)$

$\therefore \int_0^{\pi/2} \sin^p x \cos^q x dx \times \int_0^{\pi/2} \sin^q x \cos^p x dx$

using Result $\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\sqrt{\pi} \cdot \Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{p+q+2}{2})}$

You will get Answer

(*) Show that $\int_0^1 \left[x \log\left(\frac{1}{x}\right) \right]^{\frac{1}{3}} dx = \left(\frac{3}{4}\right)^{\frac{4}{3}} \sqrt[3]{\frac{4}{3}}$

sol.

put $\log \frac{1}{x} = u$ or $x = e^{-u}$

$dx = -e^{-u} du$

$\therefore \int_0^1 \left(e^{-u} \right)^{\frac{1}{3}} \cdot u^{\frac{1}{3}} \cdot (-1) du$

$= \int_0^{\infty} u^{\frac{1}{3}} \cdot e^{-\frac{u}{3}} \cdot e^{-u} du = \int_0^{\infty} e^{-\frac{4u}{3}} \cdot u^{\frac{1}{3}} du$

Again put $\frac{4}{3}u = t$, $u = \frac{3}{4}t$

$du = \frac{3}{4} dt$

$= \int_0^{\infty} \left(\frac{3}{4}t \right)^{\frac{1}{3}} \cdot e^{-t} \cdot \frac{3}{4} dt = \left(\frac{3}{4} \right)^{\frac{4}{3}} \int_0^{\infty} t^{\frac{1}{3}} e^{-t} dt$

$= \left(\frac{3}{4} \right)^{\frac{4}{3}} \int_0^{\infty} t^{\frac{1}{3}} e^{-t} dt = \left(\frac{3}{4} \right)^{\frac{4}{3}} \sqrt[3]{\frac{4}{3}}$

or

You can do this Q. by putting $x = e^{-u}$

Q4 Given that $\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin(n\pi)}$

Show that $\gamma(n)\gamma(n) = \frac{\pi}{\sin(n\pi)}$

Hence evaluate $\int_0^{\infty} \frac{dy}{1+y^4}$

sol. By definition of Beta function

$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

put $t = \frac{1}{1+x}$ i.e. $1+x = \frac{1}{t}$

$dx = -\frac{1}{(1+x)^2} dx$ $x = \frac{1}{t} - 1$

$\therefore \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_{\infty}^0 \left(\frac{1}{1+x} \right)^{m-1} \left[1 - \frac{1}{1+x} \right]^{n-1} \left[\frac{-1}{(1+x)^2} \right] dx$

$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\therefore B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(n, m)$

if we take $m+n=1$
 $m=1-n$

$B(n, 1-n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx$

$B(n, 1-n) = \frac{\pi}{\sin(n\pi)}$

using $\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin(n\pi)}$

$\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(1)} = \frac{\pi}{\sin(n\pi)}$ using Relation Between Beta/Gamma function

$\therefore \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$

Now consider $\int_0^{\infty} \frac{dy}{1+y^4}$

put $y = z$, $y = z^{\frac{1}{4}}$

$dy = \frac{1}{4} z^{-\frac{3}{4}} dz$

$\therefore \int_0^{\infty} \frac{1}{1+z} \cdot \frac{1}{4} z^{-\frac{3}{4}} dz = \frac{1}{4} \int_0^{\infty} \frac{z^{\frac{1}{4}-1}}{1+z} dz$

Comparing with the result given

$\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin(n\pi)}$

for $n = \frac{1}{4}$

$\therefore \frac{1}{4} \int_0^{\infty} \frac{z^{\frac{1}{4}-1}}{1+z} dz = \frac{1}{4} \cdot \frac{\pi}{\sin(\frac{\pi}{4})} = \frac{\pi}{4 \cdot \frac{1}{\sqrt{2}}} = \frac{\pi\sqrt{2}}{4}$

OR

$B(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$ As we have proved result

$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

put $m=n$, $m=n$

$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{2n}} dx = \frac{\pi}{\sin(n\pi)}$

$\therefore \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin(n\pi)} \quad (*)$

You can also show $\Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin(n\pi)}$

Eg. Multiplying both sides by n

$\Gamma(1-n)n\Gamma(n) = \frac{n\pi}{\sin(n\pi)}$

$\Gamma(1-n)\Gamma(n+1) = \frac{n\pi}{\sin(n\pi)} \quad [\Gamma(n) = \frac{\Gamma(n+1)}{n}]$

A.S. is $\Gamma(n+1) = \frac{n\Gamma(n)}{1} = \frac{1}{5} \Gamma(\frac{2}{5} + 1)$

put $x^{\frac{1}{5}} = t$

$x = t^{\frac{1}{5}}, dx = \frac{1}{5} t^{-\frac{4}{5}} dt$

$\therefore \int_0^1 \frac{t^{\frac{1}{5}-1}}{(1-t)^{\frac{1}{5}}} \cdot \frac{1}{5} t^{-\frac{4}{5}} dt$

$= \frac{1}{5} \int_0^1 t^{-\frac{3}{5}} (1-t)^{-\frac{1}{5}} dt = \frac{1}{5} B(-\frac{3}{5}+1, -\frac{1}{5}+1)$

$= \frac{1}{5} B(\frac{2}{5}, \frac{1}{2})$

Prove that $\int_0^1 \frac{dx}{\sqrt{1+x^5}} = \frac{1}{4\sqrt{2}} B(\frac{1}{4}, \frac{1}{2})$

(ii)

Sol.

put $x^{\frac{1}{5}} = \tan \theta$

$dx = \sec^2 \theta d\theta$

$\frac{dx}{\sqrt{1+x^5}} = \frac{\sec^2 \theta}{\sqrt{1+\tan^5 \theta}} d\theta$

$$\therefore \int \frac{dx}{\sqrt{1+x^2}} = \int \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \sec \theta \, d\theta$$

$$\frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^{\pi} \sqrt{2} \, d\theta = \frac{\sqrt{2}}{2} \int_0^{\pi} d\theta = \frac{\sqrt{2}}{2} \cdot \pi = \frac{\pi\sqrt{2}}{2}$$

put $2\theta = \phi \Rightarrow d\theta = \frac{d\phi}{2}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{d\phi}{2 \int \sin \phi}$$

$$= \frac{1}{2} \int_0^{\pi} (\sin \phi)^{\frac{1}{2}} \cos \phi \, d\phi$$

$$\text{using } I_2 = \int_0^{\pi} \sin^2 \theta \, d\theta = \frac{[\frac{\theta}{2} - \frac{\sin 2\theta}{4}]_0^{\pi}}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \cdot \frac{\sqrt{1} \cdot \sqrt{1}}{\sqrt{\frac{1}{2} + \frac{1}{2}}} = \frac{1}{4\sqrt{2}} \cdot \frac{\sqrt{1} \cdot \sqrt{1}}{\sqrt{1}}$$

$$= \frac{1}{25} \int_0^{\frac{\pi}{2}} 2\left(\frac{1}{5}\right)^{-1} - 2\left(\frac{1}{5}\right)^{-1} \cos \phi \, d\phi$$

$$\frac{1}{2\sqrt{3}} \cdot \frac{1}{9} \beta \left(\frac{1}{4} - \frac{1}{9} \right) = \frac{1}{4\sqrt{3}} \beta \left(\frac{1}{4} - \frac{1}{9} \right)$$

(iii) Show that $I = \int_0^1 x^2(1-\sqrt{x})^5 dx = 2/81$

Let $\sqrt{x} = y \Rightarrow x = y^2$
 $dx = 2y dy$

$$I_2 = \int_0^1 y^6 (1-y)^5 dy = 2 \int_0^1 y^7 (1-y)^5 dy = 2B(8,6) = 8.145$$

Q6 Show that $\int_a^b (b-x)^{n-1} dx = (b-a)^{m+n-1} B(m, n)$,
(i) $[m > 0, n > 0]$

Put $x = a + (b-a)z$ so that $x-a = (b-a)z$
 $dx = (b-a)dz$

$$\int_0^b (a-x)^{m-1} (b-x)^{n-1} dx = \int_0^1 (b-a)^{m-1} \cdot \{b-a-(b-a)z\}^{n-1} (b-a) dz$$

$$= \int_0^1 \int_0^1 (b-a)^{m+n-1} \cdot \frac{z^{n-1}}{(1-z)^{n-1}} \cdot \frac{(b-a)^{m-1}}{(1-z)^{m-1}} \cdot \frac{z^{p-1}}{(b-a)^{p-1}} dz$$

$$= \frac{(b-a)^{m+n-1}}{B(m,n)} = R.H.S.$$

(ii) Show that $\int_0^b (x-a)^n (b-x)^n dx = \frac{2}{n!} (b-a)^{n+1} \beta(n+1, n+1)$

Put $(x-a) = y$, $dx = dy$, $x = a+y$

$$\int_a^b (x-a)^m (b-x)^n dx = \int_y^{b-a} y^m (b-a-y)^n dy$$

Again put $y = (b-a) \sin^2 \theta$

$$dy = 2(b-a) \sin \theta \cdot \cos \theta d\theta$$

$$\therefore \int_a^b (x-a)^m (b-x)^n dx = \int_0^{\pi/2} (b-a)^m \sin^m \theta [(b-a) - (b-a) \sin^2 \theta]^n \cdot 2(b-a) \sin \theta \cos \theta d\theta$$

$$= \int_0^{\pi/2} (b-a)^{m+n+1} \sin^{2m+1} \theta \cos^{2n+1} \theta d\theta$$

$$= 2(b-a)^{m+n+1} \int_0^{\pi/2} \sin^{2m} \theta \cos^{2n} \theta d\theta$$

$$= 2(b-a)^{m+n+1} \int_0^{\pi/2} \sin^{2m+1} \theta \cos^{2n+1} \theta d\theta$$

$$= 2(b-a)^{m+n+1} \int_0^{\pi/2} \sin^{2m} \theta \cos^{2n} \theta d\theta$$

$$= \int_0^{\pi/2} \sin^{2m} \theta \cos^{2n} \theta d\theta = \beta(m, n)$$

$$= (b-a)^{m+n+1} \beta(m+1, n+1)$$

(iii) Show that $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b} \beta(m, n)$

or

$$\text{Express } \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx \text{ in terms of Beta function}$$

Let $m > 0, n > 0, a > 0, b > 0$

Sol. Put $bx = az$, $x = \frac{a}{b} z$

$$dx = \frac{a}{b} dz$$

$$\therefore \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \int_0^{\infty} \left(\frac{az}{b}\right)^{m-1} \frac{1}{(a+az)^{m+n}} \cdot \frac{a}{b} dz$$

$$= \left(\frac{a}{b}\right)^m \cdot \frac{1}{a^{m+n}} \cdot \frac{a}{b} \int_0^{\infty} \frac{z^{m-1}}{(1+z)^{m+n}} dz$$

$$= \frac{1}{a^n b} \int_0^{\infty} \frac{z^{m-1}}{(1+z)^{m+n}} dz$$

using definition of Beta function

$$= \frac{1}{a^n b} \beta(m, n)$$

(iv) Show that, for $m > 0, n > 0$, show that $\int_0^{\infty} \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx$

$$\text{Sol. } \int_0^{\infty} \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx - \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \left[\beta(m, n) - \beta(n, m) \right]$$

$$= \beta(m, n) - \beta(n, m) = \beta(m, n) - \beta(m, n) = 0$$

(v) Show that $\int_0^{\infty} \frac{x^{10} - x^{20}}{(1+x)^{30}} dx = 0$ using R. 6(iv)

Sol. $\int_0^{\infty} \frac{x^{10} - x^{20}}{(1+x)^{30}} dx = \int_0^{\infty} \frac{x^{10}}{(1+x)^{30}} dx - \int_0^{\infty} \frac{x^{20}}{(1+x)^{30}} dx$

$$= B(10, 20) - B(20, 10) = 0$$

(vi) Show that $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx = \frac{1}{4} B\left(\frac{7}{4}, \frac{1}{4}\right)$

Sol. Put $\frac{1-x^4}{1+x^4} = z$ so that $x^4 = \frac{1-z}{1+z}$
 $x = \left(\frac{1-z}{1+z}\right)^{1/4}$

$$dx = \frac{1}{4} \left(\frac{1-z}{1+z}\right)^{-3/4} \cdot \frac{(1+z)(-1) - (1-z)(1)}{(1+z)^2} dz$$

$$dx = \frac{1}{4} \left[\frac{1-z}{1+z}\right]^{-3/4} \cdot \left[\frac{-2}{(1+z)^2}\right] dz$$

$$= \frac{1}{4} \left[\frac{1+z}{1-z}\right]^{3/4} \cdot \frac{(1-z)}{(1+z)^2} dz$$

$$= -\frac{1}{4} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^2} dz$$

$$= -\frac{1}{4} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^2} dz$$

Also $1-x^4 = 1 - \frac{1-z}{1+z} = \frac{2}{1+z}$

$$1+x^4 = 1 + \frac{1-z}{1+z} = \frac{2}{1+z}$$

$$\therefore \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx = \int_0^1 \left(\frac{2}{1+z}\right)^{3/4} \cdot \frac{(1-dz)}{2(1-z)^{3/4} (1+z)^5}$$

$$= \int_0^1 \frac{2^{3/4} \cdot \left(\frac{2}{1+z}\right)^{3/4}}{2 \cdot \left(\frac{2}{1+z}\right)^2} dz = \int_0^1 \frac{2^{3/4} \cdot \left(\frac{2}{1+z}\right)^{3/4}}{(1-z)^{3/4} (1+z)^5} dz$$

$$= \frac{1}{2} \int_0^1 \frac{2^{3/4}}{z^{3/4} (1+z)^2} dz = \frac{1}{2} \int_0^1 \frac{2^{3/4}}{(1+z)^2} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^{5/4}} dz$$

$$= \frac{1}{2} \int_0^1 \frac{2^{3/4}}{(1+z)^2} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^{5/4}} dz$$

$$= \frac{1}{2} \int_0^1 \frac{2^{3/4}}{(1+z)^2} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^{5/4}} dz$$

$$= \frac{1}{4} \int_0^1 \frac{2^{3/4}}{(1+z)^2} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^{5/4}} dz$$

$$= \frac{1}{4} \int_0^1 \frac{2^{3/4}}{(1+z)^2} \cdot \frac{1}{(1-z)^{3/4}} \cdot \frac{1}{(1+z)^{5/4}} dz$$

Q7. Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$

where n is a positive integer and $m > -1$
Hence evaluate $\int_0^1 x (\log x)^3 dx$

Sol. We have $I = \int_0^1 x^m (\log x)^n dx$

Put $\log x = t$, $x = e^t$

$dx = e^t dt$

$$I = \int_{-\infty}^0 e^{mt} \cdot t^n \cdot e^t dt = \int_{-\infty}^0 e^{(m+1)t} \cdot t^n dt$$

Again put $(m+1)t = -z$
 $dt = \frac{-1}{(m+1)} dz$

$$I = \int_{-\infty}^0 e^{-z} \cdot \frac{(-z)^n}{(m+1)} \cdot \frac{(-1)}{(m+1)} dz$$

$$= (-1)^n \int_0^{\infty} z^n \cdot e^{-z} dz = \frac{(-1)^n}{(m+1)^{n+1}} \cdot \Gamma(n+1)$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}} \quad \text{[L.H.S.]} \quad \left[\Gamma(n+1) = n! \right]$$

Now $\int_0^1 x (\log x)^3 dx = \frac{(-1)^3 \cdot 3!}{(1+1)^{3+1}} = \frac{-6}{16} = -\frac{3}{8}$ Ans

Q8. Show that $\int_0^1 y^{q-1} \left(\log \frac{1}{y} \right)^{p-1} dy = \frac{\Gamma p}{q^p}$, $p > 0, q > 0$

Sol.

L.H.S. $= \int_0^1 y^{q-1} \left(\log \frac{1}{y} \right)^{p-1} dy$

Put $\log \left(\frac{1}{y} \right) = t$, $\frac{1}{y} = e^t$

$y = e^{-t}$, $dy = -e^{-t} dt$

L.H.S. $= \int_{-\infty}^0 \left(e^{-t} \right)^{q-1} \cdot t^{p-1} \cdot e^{-t} (-dt)$

$$= \int_0^{\infty} t^{p-1} \cdot e^{-(q-1)t} \cdot e^{-t} dt$$

$$= \int_0^{\infty} e^{-(q-1)t} t^{p-1} dt$$

$$= \int_0^{\infty} e^{-qt} t^{p-1} dt$$

put $qt = m$, $t = \frac{m}{q}$
 $q dt = dm$

L.H.S. $= \int_0^{\infty} e^{-m} \cdot \left(\frac{m}{q} \right)^{p-1} \cdot \frac{dm}{q}$

$$= \frac{1}{q^p} \int_0^{\infty} e^{-m} \cdot m^{p-1} dm = \frac{1}{q^p} \Gamma(p) = \text{R.H.S.}$$

Express $\int_0^1 x^m (1-x)^n dx$ in terms of Gamma function.

(i) $\int_0^1 x^5 (1-x)^3 dx$ and (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol.

$$\text{Let } I = \int_0^1 x^m (1-x^n)^p dx$$

$$\text{Put } x^n = y \Rightarrow x = y^{\frac{1}{n}}$$

$$dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$\therefore I = \int_0^1 y^{\frac{m}{n}} (1-y)^p \cdot \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$= \frac{1}{n} \int_0^1 y^{\frac{m}{n} + \frac{1}{n} - 1} (1-y)^p dy$$

$$= \frac{1}{n} \int_0^1 y^{\frac{m+n}{n}-1} (1-y)^p dy$$

$$= \frac{1}{n} \cdot B\left[\frac{m+1}{n}, p+1\right] = \frac{1}{n} \cdot \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p+1\right)}$$

$$\text{using } B(m,n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\text{Now calculate } \int_0^1 x^5 (1-x^3)^{10} dx$$

$$\text{and comparing with } \int_0^1 x^m (1-x^n)^p dx$$

$$\text{so using result } \textcircled{*} \text{ where } m=5, n=3, p=10$$

$$\int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{3} \cdot \frac{\Gamma\left(\frac{5+1}{3}\right) \Gamma(10+1)}{\Gamma\left(\frac{5+1}{3} + (10+1)\right)}$$

$$= \frac{1}{3} \cdot \frac{\Gamma(2) \Gamma(11)}{\Gamma(13)} = \frac{1}{3} \cdot \frac{1! \cdot 10!}{12!} = \frac{1}{396} \text{ Ans}$$

(ii)

$$\int_0^1 \frac{dx}{1-x^n} = \int_0^1 x^0 (1-x^n)^{-1} dx$$

$$\text{comparing with } \int_0^1 x^m (1-x^n)^p dx$$

$$\text{where } m=0, n=n, p=-\frac{1}{n}$$

$$\text{using result } \textcircled{*}$$

$$\int_0^1 x^0 (1-x^n)^{-\frac{1}{n}} dx = \frac{1}{n} \cdot \frac{\Gamma\left(\frac{0+1}{n}\right) \Gamma\left(-\frac{1}{n} + 1\right)}{\Gamma\left(\frac{0+1}{n} + \left(-\frac{1}{n} + 1\right)\right)}$$

$$= \frac{1}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(1 - \frac{1}{n}\right)}{\Gamma(1)} = \frac{\sqrt[n]{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(n \cdot \frac{1}{n}\right)}$$

$$\text{Q10 Show that } \int_0^1 (1+x)^{p-1} (1-x)^{q-1} dx = 2^{p+q-1} B(p,q)$$

$$\text{for put } 1+x = 2t, dx = 2dt$$

$$\text{when } x=1, t=1 \text{ and when } x=-1, t=0$$

$$\therefore \int_0^1 (1+x)^{p-1} (1-x)^{q-1} dx = \int_0^1 (2t)^{p-1} [1-(2t-1)]^{q-1} \cdot 2dt$$

$$= \int_0^1 (2t)^{p-1} (2-2t)^{q-1} \cdot 2dt = 2^{p-1} \cdot 2^{q-1} \cdot 2 \int_0^1 t^{p-1} (1-t)^{q-1} dt$$

$$= 2^{p+q-1} B(p,q)$$

Q 11

Prove that $\int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$
and hence evaluate $\int_0^{\infty} \operatorname{sech}^8 x \, dx$

Sol.

$$\text{Let } I = \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n}$$

put $x = \tan \theta$, $e^x = \sec^2 \theta$

$$dx = \frac{\sec^2 \theta \cdot \sec^2 \theta}{\sec^4 \theta} = \frac{\sec^2 \theta}{\tan^2 \theta}$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{[(\tan \theta) + (\tan \theta)^{-1}]^n} \cdot \frac{\sec^2 \theta}{\tan^2 \theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(\tan \theta)^n}{[(\tan \theta)^2 + 1]^n} \cdot \frac{\sec^2 \theta}{\tan^2 \theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\tan \theta)^{n-1} \cdot \frac{\sec^2 \theta}{(\sec^2 \theta)^n} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\tan \theta)^{n-1} (\sec^2 \theta)^{1-n} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\tan \theta)^{n-1} (\cos^2 \theta)^{n-1} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin \theta)^{n-1} (\cos \theta)^{n-1} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta$$

Now

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta - \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{n-1} \theta \cos^{n-1} \theta d\theta$$

$$= \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right) - \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right) - \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right) - \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{sech}^8 x = \frac{2^8}{(e^x + e^{-x})^8}$$

$$\therefore \int_0^{\infty} \operatorname{sech}^8 x \, dx = \int_0^{\infty} \frac{2^8}{(e^x + e^{-x})^8} dx$$

$$\text{using result } \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

from (8)

$$\int_0^{\infty} \operatorname{sech}^8 x \, dx = \frac{2^8}{4} B\left[\frac{8}{2}, \frac{8}{2}\right] = 2^6 B(4, 4)$$

$$= \frac{2^6 \cdot 14! \cdot 14!}{7! \cdot 7!} = \frac{2^6 \cdot 31 \times 3!}{35 \cdot 35} = \frac{16}{35} \text{ Ans}$$

Q. 12

Prove that $B\left[n+\frac{1}{2}, n+\frac{1}{2}\right] = \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}} \sqrt{\pi}$

Hence show that $\sqrt{n} \left[\sqrt{n+\frac{1}{2}} \right] = 1.3.5 \dots (2n-1) \sqrt{\pi}$

Sol.

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \sin^{\frac{2(n+\frac{1}{2})-1}{2}} \cos^{\frac{2(n+\frac{1}{2})-1}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n} \cos^{2n} d\theta$$

$$\text{Using } B(m, n) = \int_0^{\frac{\pi}{2}} \sin^{2m-1} \cos^{2n-1} d\theta$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} (\sin \theta \cos \theta)^{2n} d\theta = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2n} d\phi$$

put $\theta = \phi$, $d\theta = d\phi$

when $\theta = 0 \Rightarrow \phi = 0$ and $\theta = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{2}$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2n} d\phi = \frac{1}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2n} d\phi$$

$$\text{Using } \int_0^{\frac{\pi}{2}} \sin^m x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right)}{2 \Gamma\left(\frac{m+2}{2}\right)}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \frac{\left[\frac{2n+1}{2} \right]}{\left[\frac{2n+2}{2} \right]} = \frac{\sqrt{\pi}}{\sqrt{n}} \frac{\left[n+\frac{1}{2} \right]}{\left[n+1 \right]} \quad \text{--- (1)}$$

Using Relation $B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\left[\frac{2n+1}{2} \right]^2}{\left[\frac{2n+1}{2} + \frac{2n+1}{2} \right]} = \frac{\left[\frac{2n+1}{2} \right]^2}{\left[2n+1 \right]} \quad \text{--- (2)}$$

From (1) and (2)

$$\frac{\left[\frac{2n+1}{2} \right]^2}{\left[2n+1 \right]} = \frac{\sqrt{\pi}}{\sqrt{n}} \cdot \frac{\left[n+\frac{1}{2} \right]}{\left[n+1 \right]}$$

$$\sqrt{n} \left[n+\frac{1}{2} \right] = \frac{\sqrt{\pi}}{\left[n+1 \right]} \left[\frac{2n+1}{2} \right]^2$$

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$$= \frac{\sqrt{\pi}}{1} \cdot \frac{2n \cdot 2n-1 \cdot 2n-2 \cdot 2n-3 \cdot 2n-4 \dots}{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4 \dots}$$

$$= \frac{\sqrt{\pi}}{1} \cdot \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-1) \cdot 2n}{n \cdot n-1 \cdot n-2 \cdot n-3 \dots 1}$$

$$= \frac{\sqrt{\pi}}{1} \cdot \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-1) \cdot 2n}{1}$$

$$\sqrt{n} \left[n+\frac{1}{2} \right] = \sqrt{\pi} \{ 1 \cdot 3 \cdot 5 \dots 2n-1 \}$$

$$2 \cdot \sqrt{n+1} = \{1 \cdot 3 \cdot 5 \cdots (2n-1)\} \sqrt{\pi}$$

13 Show that

$$(i) \frac{B(m+1, n)}{B(m, n)} = \frac{m}{m+n} \quad \text{or} \quad \frac{B(m+1, n)}{B(m, n)} = \frac{B(m, n)}{m+n}$$

Sol. Using $B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$ or $B(m+1, n) = \frac{B(m, n)}{m+n}$

$$\therefore \frac{B(m+1, n)}{B(m, n)} = \frac{\Gamma(m+1) \Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(m+n)}{\Gamma(m) \Gamma(n)}$$

$$= \frac{m \Gamma(m) \Gamma(n)}{(m+n) \Gamma(m+n)} = \frac{m}{m+n}$$

(ii) Show that $B(n, n) = \frac{\sqrt{\pi} \Gamma(n)}{2^{2n-1} \Gamma(n+\frac{1}{2})}$

Sol. By definition of Beta function, we have

$$B(m, n) = \int_0^1 \int_0^{1-x} x^{m-1} y^{n-1} dx dy$$

$$\therefore B(n, n) = \int_0^1 \int_0^{1-x} x^{n-1} y^{n-1} dx dy$$

$$= \int_0^1 \int_0^{1-x} x^{n-1} y^{n-1} dx dy$$

$$= \frac{1}{2^{2n-1}} \int_0^1 (1-x)^{2n-1} dx$$

put $2\theta = z$, $d\theta = \frac{dz}{2}$, $\theta = \frac{\pi}{2} \Rightarrow z = \pi$
 $\theta = 0 \Rightarrow z = 0$

$$B(n, n) = \frac{1}{2^{2n-1}} \int_0^\pi (\sin z)^{2n-1} \frac{dz}{2}$$

$$= \frac{1}{2^{2n}} \int_0^\pi (\sin z)^{2n-1} dz$$

$$= \frac{1}{2^{2n-1}} \int_0^\pi (\sin z)^{2n-1} dz$$

using $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_a^{2a} f(x) dx$
 $\int_0^\pi f(x) dx = \int_0^a f(x) dx + \int_a^\pi f(x) dx$

$$= \frac{1}{2^{2n-1}} \left[\int_0^{\frac{\pi}{2}} (\sin z)^{2n-1} dz + \int_{\frac{\pi}{2}}^\pi (\sin z)^{2n-1} dz \right]$$

using $\int_0^\pi f(x) dx = \int_0^\pi f(\pi-x) dx$
 $\int_0^\pi f(x) dx = \int_0^\pi f(\pi-x) dx$

$$= \frac{1}{2^{2n-1}} \left[\int_0^{\frac{\pi}{2}} (\sin z)^{2n-1} dz + \int_0^{\frac{\pi}{2}} (\sin(\pi-z))^{2n-1} dz \right]$$

$$= \frac{1}{2^{2n-1}} \left[\int_0^{\frac{\pi}{2}} (\sin z)^{2n-1} dz + \int_0^{\frac{\pi}{2}} (\sin z)^{2n-1} dz \right]$$

(iii) Show that $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n}} \Gamma(n+1)$

Sol.

using definition of Beta function

$$B(m, n) = \int_0^1 \int_0^{1-x} x^{m-1} y^{n-1} dx dy$$

$$\therefore B(n+\frac{1}{2}, n+\frac{1}{2}) = \int_0^1 \int_0^{1-x} x^{n+\frac{1}{2}-1} y^{n+\frac{1}{2}-1} dx dy$$

$$= \int_0^{\pi} \sin^n \theta \cdot \cos \theta \, d\theta = \int_0^{\pi} \left[\frac{\sin^{n+1} \theta}{n+1} \right]_0^{\pi} d\theta$$

$$= \frac{1}{2^n} \int_0^{\pi} (\sin \theta)^{2n} d\theta$$

$$\text{put } \sin \theta = t, \quad d\theta = \frac{dt}{\cos \theta}$$

$$\text{when } \theta = 0 \Rightarrow t = 0$$

$$\theta = \pi \Rightarrow t = -1$$

$$B\left(n+1, n+1\right) = \frac{1}{2^n} \int_0^{\pi} (\sin \theta)^{2n} d\theta = \frac{1}{2^n} \int_0^{\pi} (\sin \theta)^{2n} d\theta$$

$$= \frac{1}{2^n} \int_0^{\pi} (\sin \theta)^{2n} d\theta$$

$$\text{using } \int_0^{\pi} f(\sin \theta) d\theta = 2 \int_0^{\pi/2} f(\sin \theta) d\theta$$

$$= \frac{1}{2^n} \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{2^n} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \quad \text{--- (*)}$$

Using Relation B/Gamma

$$B\left(n+1, n+1\right) = \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} + \frac{n+1}{2}\right)} = \frac{\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}{\Gamma(n+1)} \quad \text{--- (**)}$$

From Eqⁿ (x) and (**)

$$\frac{\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2^n} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 = \frac{\sqrt{\pi}}{2^n} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+1}{2}\right)}$$

Show that $B(m+1, n) + B(m, n+1) = B(m, n)$

Sol. L.H.S is $B(m+1, n) + B(m, n+1)$

$$= \int_0^1 x^{m+1-1} \cdot (1-x)^{n-1} dx + \int_0^1 x^{m-1} (1-x)^{n+1-1} dx$$

$$= \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1} (1-x)^n dx$$

$$= \int_0^1 x^m (1-x)^{n-1} + x^{m-1} (1-x)^n dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} [x + 1-x] dx$$

$$= \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx = B(m, n) = R.H.S$$

Show that $B(m+1, n) = \frac{B(m, n)}{n}$ and $B(m, n+1) = \frac{B(m, n)}{m+1}$

$$\frac{B(m+1, n)}{m} = \frac{\Gamma(m+1) \cdot \Gamma(n)}{\Gamma(m+n+1)} = \frac{m! \cdot \Gamma(n)}{(m+1)! \cdot \Gamma(m+n+1)} = \frac{\Gamma(m) \cdot \Gamma(n)}{m! \cdot \Gamma(m+n+1)}$$

$$\frac{B(m+1, n)}{m} = \frac{n \sqrt{m} \sqrt{n}}{n \sqrt{m+n+1}} = \frac{\sqrt{m} \sqrt{n+1}}{\sqrt{m+n+1}}$$

$$\frac{B(m+1, n)}{m} = \frac{B(m, n+1)}{n} \quad \text{--- (1)}$$

$$\text{Also } \frac{B(m+1, n)}{m} = \frac{\sqrt{m+1} \cdot \sqrt{n}}{m \sqrt{m+1+n}} = \frac{\sqrt{m} \sqrt{n+1}}{m \sqrt{m+1+n}}$$

$$= \frac{\sqrt{m} \sqrt{n}}{(m+n) \sqrt{m+n}} = \frac{B(m, n)}{(m+n)} \quad \text{--- (11)}$$

from (I) and (II)

$$\frac{B(m+1, n)}{m} = \frac{B(m, n+1)}{n} = \frac{B(m, n)}{m+n}$$

Some Extra Questions

1. Evaluate $\int_0^{\infty} e^{-x^3} dx$

Sol. Put $x^3 = z$ or $x = z^{\frac{1}{3}}$

so that $dx = \frac{1}{3} z^{-\frac{2}{3}} dz$

$$I = \int_0^{\infty} e^{-z} \cdot \frac{1}{3} z^{-\frac{2}{3}} dz = \frac{1}{3} \int_0^{\infty} e^{-z} z^{-\frac{2}{3}-1} dz = \frac{1}{3} \Gamma^{\frac{1}{3}}$$

$$I = \frac{1}{3} \Gamma^{\frac{1}{3}} = \frac{\sqrt{4}}{3} \quad [\because (n-1) \Gamma(n-1) = \Gamma n]$$

2. Evaluate $I = \int_0^{\infty} x^3 e^{-x^3} dx$

Sol. put $x^3 = z$, $x = z^{\frac{1}{3}}$ so that $dx = \frac{1}{3} z^{-\frac{2}{3}} dz$

$$I = \int_0^{\infty} z e^{-z} \cdot \frac{1}{3} z^{-\frac{2}{3}} dz = \frac{1}{3} \int_0^{\infty} z^{\frac{1}{3}-1} e^{-z} dz$$

$$= \frac{1}{3} \int_0^{\infty} e^{-z} z^{\frac{1}{3}-1} dz = \frac{1}{3} \Gamma^{\frac{4}{3}}$$

$$= \frac{1}{3} \cdot \frac{1}{3} \Gamma^{\frac{1}{3}} = \frac{1}{9} \Gamma^{\frac{1}{3}}$$

using $\Gamma n = (n-1) \Gamma(n-1)$

3. Evaluate $I = \int_0^{\infty} e^{-a^3 x^3} dx$, ($a > 0$)

Sol. Put $a^3 x^3 = z$, $x = \frac{\sqrt[3]{z}}{a}$ so that $dx = \frac{1}{a} z^{-\frac{2}{3}} dz$

$$I = \int_0^{\infty} e^{-z} \cdot \frac{1}{a} z^{-\frac{2}{3}} dz = \frac{1}{a} \int_0^{\infty} e^{-z} z^{\frac{1}{3}-1} dz$$

$$= \frac{1}{a} \Gamma^{\frac{4}{3}} = \frac{\sqrt{\pi}}{a}$$

$$\therefore \Gamma^{\frac{1}{3}} = \sqrt{\pi}$$

4. Prove that if $n > -1$, $\int_0^{\infty} x^n e^{-a^3 x^3} dx = \frac{1}{3a^{n+1}} \Gamma^{\frac{n+1}{3}}$

Hence or otherwise show that $\int_{-\infty}^{\infty} e^{-a^3 x^3} dx = \frac{\sqrt{\pi}}{a}$

Sol. Put $a^3 x^3 = z$ i.e. $x = \frac{\sqrt[3]{z}}{a}$ so that $dx = \frac{1}{3a} z^{-\frac{2}{3}} dz$

$$I = \int_0^{\infty} x^n e^{-a^3 x^3} dx = \int_0^{\infty} \frac{z^{\frac{n}{3}}}{a^n} \cdot e^{-z} \cdot \frac{z^{-\frac{2}{3}}}{3a} dz$$

$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} e^{-z} z^n dz = \frac{1}{\Gamma(n+1)} \int_0^{\infty} e^{-z} z^{n-1+1} dz$$

$$I = \frac{1}{\Gamma(n+1)} \int_0^{\infty} e^{-z} z^n dz, \text{ Hence proved}$$

Putting $n=0$

$$I = \int_0^{\infty} x^0 \cdot e^{-x^2} dx = \int_0^{\infty} e^{-x^2} dx$$

$$\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx$$

as in Question in (3)

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx$$

$$= 2 \left[\frac{1}{2a} \cdot \frac{1}{2} \right]$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is even function}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{a} \sqrt{\pi} = \sqrt{\pi} \text{, hence proved.}$$

Q.5 Show that $\int_0^{\infty} \frac{x^a}{e^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$, $a > -1$

Sol. $a^x = e^{x \log a} = e^{-x \log a}$

$$I = \int_0^{\infty} \frac{x^a}{e^{x \log a}} dx = \int_0^{\infty} e^{-x \log a} \cdot x^a dx$$

Put $x \log a = z$, $dx = \frac{dz}{\log a}$

$$I = \int_0^{\infty} e^{-z} \cdot \left(\frac{z}{\log a} \right)^a \cdot \frac{dz}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-z} z^a dz$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-z} z^{a+1-1} dz$$

$$= \frac{1}{(\log a)^{a+1}} \cdot \frac{\Gamma(a+1)}{\Gamma(a+1)}$$

Q.6 Show that $\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{1}{a^n}$, where a, n are positive.

Hence show that

$$(i) \int_0^{\infty} e^{-ax} x^{n-1} \cos bx dx = \frac{\Gamma(n)}{a^n} \cos \theta$$

$$(ii) \int_0^{\infty} e^{-ax} x^{n-1} \sin bx dx = \frac{\Gamma(n)}{a^n} \sin \theta$$

where $\theta = \tan^{-1} \frac{b}{a}$ and $\theta = \tan^{-1} \frac{b}{a}$

Sol. Put $ax = z$ so that $dx = \frac{dz}{a}$

When $x=0, z=0$, when $x \rightarrow \infty, z \rightarrow \infty$

$$\therefore \int_0^{\infty} e^{-ax} x^{n-1} dx = \int_0^{\infty} e^{-z} \cdot \left(\frac{z}{a} \right)^{n-1} \cdot \frac{dz}{a}$$

$$= \frac{1}{a^n} \int_0^{\infty} e^{-z} z^{n-1} dz = \frac{\Gamma(n)}{a^n} \text{, Hence proved}$$

Replacing a by $a+ib$, we have

$$\int_0^{\infty} e^{-(a+ib)x} \cdot x^{n-1} dx = \frac{\Gamma(n)}{(a+ib)^n} \quad \text{--- (1)}$$

Now $e^{-(a+ib)x} = e^{-ax} \cdot e^{-ibx}$

$$= e^{-ax} [\cos bx - i \sin bx]$$

Also putting $a = r \cos \theta$ and $b = r \sin \theta$

$$a^2 + b^2 = r^2 \quad \text{and} \quad \frac{b}{a} = \tan \theta \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

and $(a+ib)^n = (r \cos \theta + i r \sin \theta)^n = r^n (\cos \theta + i \sin \theta)^n$
 $= r^n (\cos n\theta + i \sin n\theta)$

From (1),

$$\int_0^{\infty} e^{-ax} [\cos bx - i \sin bx] \cdot x^{n-1} dx = \frac{\Gamma(n)}{r^n (\cos n\theta + i \sin n\theta)}$$

$$= \frac{\Gamma(n)}{r^n} \cdot [\cos n\theta - i \sin n\theta]$$

$$= \frac{\Gamma(n)}{r^n} [\cos n\theta - i \sin n\theta]$$

Comparing Real Parts And Imaginary Parts

$$\int_0^{\infty} e^{-ax} \cdot x^{n-1} \cdot \cos bx dx = \frac{\Gamma(n)}{r^n} \cdot \cos n\theta$$

$$\int_0^{\infty} e^{-ax} \cdot x^{n-1} \cdot \sin bx dx = \frac{\Gamma(n)}{r^n} \cdot \sin n\theta$$

Q.7

(i)

Evaluate $\int_0^{\pi/2} \sin^3 x \cdot \cos^5 x dx$

using $\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{1}{2} \cdot \frac{\Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{3+1}{2}) \cdot \Gamma(\frac{5+1}{2})}{\Gamma(\frac{3+1}{2} + \frac{5+1}{2})} = \frac{1}{2} \cdot \frac{\Gamma(2) \cdot \Gamma(3)}{\Gamma(5)}$$

$$= \frac{1}{2} \cdot \frac{1 \cdot \Gamma(2)}{\frac{4}{3} \cdot \frac{3}{2} \cdot \Gamma(2)} = \frac{8}{72}$$

(ii) $\int_0^{\pi/2} \sin^7 x dx = \int_0^{\pi/2} \sin^6 x \cdot \cos x dx = \frac{1}{2} \cdot \frac{\Gamma(4) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})}$

$$= \frac{1}{2} \cdot \frac{3 \sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} = \frac{6}{105} = \frac{16}{35}$$

(iii) $\int_0^{\pi/2} \tan \theta d\theta = \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta} d\theta = \int_0^{\pi/2} (\sin \theta)^{-1} \cdot (\cos \theta)^{-2} d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} + \frac{1}{2})} = \frac{1}{2} \cdot \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(2)}$$

$$= \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{1} = \frac{\pi}{4} \cdot \sqrt{2} \pi = \frac{\pi}{\sqrt{2}} \quad \left| \because \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2}) = \sqrt{2} \pi \right|$$

(iv) Show that $\int_0^{\pi/2} \sin^m \theta d\theta = \int_0^{\pi/2} \sin^m \theta \cdot \cos^0 \theta d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{m+1}{2}) \cdot \Gamma(\frac{0+1}{2})}{\Gamma(\frac{m+1}{2} + \frac{0+1}{2})} = \frac{1}{2} \cdot \frac{\Gamma(\frac{m+1}{2}) \cdot \sqrt{\pi}}{\Gamma(\frac{m+3}{2})} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+3}{2})}$$

Hence proved

Q8.

show that $\int_0^1 \frac{1}{\sqrt{1-x^4}} dx = \frac{1}{8} \sqrt{\frac{2}{\pi}} \left[\Gamma\left(\frac{1}{4}\right) \right]^2$

Sol. Let $I = \int_0^1 \frac{1}{\sqrt{1-x^4}} dx$

Put $x = z^{\frac{1}{4}}$ i.e. $x = z^{\frac{1}{4}}$ so that $dx = \frac{1}{4} z^{-\frac{3}{4}} dz$

$$I = \int_0^1 \frac{1}{\sqrt{1-z}} \cdot \frac{1}{4} z^{-\frac{3}{4}} dz$$

$$= \frac{1}{4} \int_0^1 z^{-\frac{3}{4}} (1-z)^{-\frac{1}{2}} dz = \frac{1}{4} B\left[\frac{1}{4}, \frac{1}{2}\right]$$

using $B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

$$= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}\right)} = \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4} + \frac{1}{4}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma(1)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{1}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma(1)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{1} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{1}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma(1)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{1} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)^2}{1}$$

Q9.

Prove that $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

Sol. $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = B(m, n)$

$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

using Relation B/G
Beta & Gamma
function

Q10 Evaluate $\int_0^1 x^3 (1-x)^{1/3} dx$

Sol.

Let $I = \int_0^1 x^3 (1-x)^{1/3} dx$

put $x = \sin^2 \theta$: $dx = 2 \sin \theta \cos \theta d\theta$

$$I = \int_0^{\pi/2} \sin^6 \theta \cdot \cos^{8/3} \theta \cdot 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^7 \theta \cdot \cos^{11/3} \theta d\theta$$

$$= 2 \cdot \frac{1}{2} \frac{\left[\frac{8}{3}\right] \left[\frac{11}{3} + 1\right]}{\left(\frac{8}{3} + \frac{11}{3} + 1\right)} = \frac{4 \cdot \left[\frac{7}{3}\right]}{\left[\frac{19}{3}\right]}$$

$$= \frac{13 \cdot \left[\frac{7}{3}\right]}{\left[\frac{19}{3}\right]} = \frac{6 \times 81}{16 \cdot 13 \cdot 10 \cdot 7} = \frac{243}{7280}$$

Q17 show that $\int_0^1 \left(\frac{n+1}{x}\right) = 1 \cdot 3 \cdot 5 \dots (2n-1) \pi$, where n is positive integer.

Sol. $\int_0^1 \left(\frac{n+1}{x}\right) = \left(\frac{n-1}{2}\right) \left(\frac{n-3}{2}\right) \left(\frac{n-5}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{\pi}{2}}$

using $\Gamma n = n-1 \cdot n-2 \cdot n-3 \dots 1$
 $= n-1 \cdot n-2 \dots 3 \cdot 2 \cdot 1$

$$\frac{n+1}{2} = \left(\frac{2n-1}{2}\right) \left(\frac{2n-3}{2}\right) \left(\frac{2n-5}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$= \frac{1}{2^n} (2n-1)(2n-3) \dots 3 \cdot 1 \sqrt{\pi}$$

Listing the factors in successive orders

$$3^n \left(n + \frac{1}{3} \right) = 1 \cdot 3 \cdot 5 \cdots (3n-1) \sqrt{\pi}$$

Q.18 Show that $\int_0^1 \sqrt{1-x^4} dx = \frac{1}{12} \sqrt{\pi} \left[\Gamma\left(\frac{1}{4}\right) \right]^2$

Sol. Let $I = \int_0^1 \sqrt{1-x^4} dx$

put $x^4 = z$, i.e. $x = z^{\frac{1}{4}}$

$dx = \frac{1}{4} z^{-\frac{3}{4}} dz$

$$I = \int_0^1 (1-z)^{\frac{1}{2}} \cdot \frac{1}{4} z^{-\frac{3}{4}} dz = \frac{1}{4} \int_0^1 z^{-\frac{3}{4}} (1-z)^{\frac{1}{2}} dz$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right)$$

using def of Beta fn

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{3}{2}\right)} = \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{4} \cdot \frac{\frac{1}{2} \sqrt{\pi} \cdot \Gamma\left(\frac{1}{4}\right)}{\frac{3}{4} \Gamma\left(\frac{3}{4}\right)} = \frac{1}{6} \sqrt{\pi} \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{1}{6} \sqrt{\pi} \left[\Gamma\left(\frac{1}{4}\right) \right]^2$$

using $\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \sqrt{3}\pi$

$$= \frac{1}{6\sqrt{3}\sqrt{\pi}} \cdot \left[\Gamma\left(\frac{1}{4}\right) \right]^2$$

$$I = \frac{1}{12} \sqrt{\frac{2}{\pi}} \left[\Gamma\left(\frac{1}{4}\right) \right]^2$$

Hence proved.

Q.19 (i) Evaluate the following integrals

$$\int_0^{\infty} x^6 \cdot e^{-2x} dx$$

put $2x = z$, $dx = \frac{dz}{2}$

$$\therefore \int_0^{\infty} \left(\frac{z}{2}\right)^6 \cdot e^{-z} \cdot \frac{dz}{2} = \frac{1}{128} \int_0^{\infty} z^6 \cdot e^{-z} dz = \frac{1}{128} \Gamma(7)$$

$$= \frac{1}{128} \times (6!) = \frac{45}{8}$$

(ii) $I = \int_0^9 \frac{x^2}{\sqrt{2-x}} dx$

put $x = 2z$, $dx = 2dz$

$$I = \int_0^9 \frac{(2z)^2}{\sqrt{2-2z}} \cdot 2dz = 4\sqrt{2} \int_0^{\frac{9}{2}} z^2 (1-z)^{-\frac{1}{2}} dz$$

$$= 4\sqrt{2} B\left(3, \frac{1}{2}\right) = 4\sqrt{2} \cdot \frac{\Gamma(3) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3 + \frac{1}{2}\right)} = 4\sqrt{2} \cdot \frac{2! \cdot \frac{1}{2} \sqrt{\pi}}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{8\sqrt{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \sqrt{\pi}}{15} = \frac{64\sqrt{2}\sqrt{\pi}}{15}$$

(iii) $I = \int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \int_0^3 \frac{dx}{\sqrt{3-x}}$

put $x = 3z$, $dx = 3dz$

$$\therefore I = \int_0^1 \frac{3dz}{\sqrt{3-3z}} = \int_0^1 \frac{1}{\sqrt{1-z}} dz = \left[-2\sqrt{1-z} \right]_0^1 = 2$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\sqrt{\pi}}{\Gamma 1} = \pi \text{ Ans } [\because \Gamma 1 = 1]$$

$$(iv) I = \int_0^{\infty} \frac{x^8 (1-x^6)}{(1+x)^{24}} dx$$

$$I = \int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}} dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$= B(9, 15) - B(15, 9) = 0$$

$$\therefore \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m, n)$$

$$(v) I = \int_0^{\infty} \frac{x^4 (1+x^5)}{(1+x)^{15}} dx$$

$$= \int_0^{\infty} \frac{x^4}{(1+x)^{15}} dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}} dx$$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}} dx$$

$$= B(5, 10) + B(10, 5) = 2B(5, 10)$$

$$= 2 \frac{\Gamma 5 \Gamma 10}{\Gamma 15} = 2 \frac{14 \cdot 14}{114} = \frac{1}{5005}$$

$$Q 19 \text{ Show that } B(m, m) = 2^{-2m} B(m, \frac{1}{2})$$

Q 20 By definition of Beta function

$$B(m, m) = \int_0^{\pi/2} \sin^{2m-2} \theta \cos^{2m-2} \theta \cdot 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta$$

$$= 2 \int_0^{\pi/2} \left(\frac{2 \sin \theta \cos \theta}{2} \right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta$$

$$\text{put } 2\theta = \phi \Rightarrow d\theta = \frac{d\phi}{2}$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} \frac{\phi}{2} \cos^{2m-1} \frac{\phi}{2} \cdot \frac{1}{2} d\phi = \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} \phi \cos^{2m-1} \phi d\phi$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} \phi \cos^{2m-1} \phi d\phi \quad \left[\because \int_0^{\pi} f(\sin \phi, \cos \phi) d\phi = 2 \int_0^{\pi/2} f(\sin \phi, \cos \phi) d\phi \right]$$

$$B(m, m) = \frac{1}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} \phi \cos^{2m-1} \phi d\phi$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma\left(\frac{2m-1}{2}\right) \Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(2m-1)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma\left(\frac{2m-1}{2}\right) \Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(2m-1)}$$

$$B(m, m) = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma\left(\frac{2m-1}{2}\right) \Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(2m-1)} = 2^{-2m} B(m, \frac{1}{2})$$

Hence proved.

Q.20

show that $B(m, m) \cdot B\left(m + \frac{1}{g}, m + \frac{1}{g}\right) = \frac{\pi m^{-1}}{g^{4m-1}}$

Sol.

L.H.S

$$B(m, m) \cdot B\left(m + \frac{1}{g}, m + \frac{1}{g}\right)$$

$$= \frac{\Gamma_m \cdot \Gamma_m}{\Gamma_{gm}} \cdot \frac{\Gamma_{m+\frac{1}{g}} \cdot \Gamma_{m+\frac{1}{g}}}{\Gamma_{(gm+1)}} \quad \left\{ \begin{array}{l} \text{using} \\ B(m, n) = \frac{\Gamma_m \cdot \Gamma_n}{\Gamma_{m+n}} \end{array} \right.$$

$$= \left[\Gamma_m \cdot \Gamma_{\left(m+\frac{1}{g}\right)} \right]^2 \cdot \frac{1}{\Gamma_{gm} \cdot \Gamma_{gm}} \left[\Gamma_m \cdot \Gamma_{\left(m+\frac{1}{g}\right)} \right]^2 \quad \text{--- (1)}$$

using duplication formula [Example No. 7.47]

$$\Gamma_m \cdot \Gamma_{m+\frac{1}{g}} = \frac{\sqrt{\pi}}{g^{m-1}} \cdot \Gamma_{gm}$$

$$\begin{aligned} & \text{from (1)} \\ &= \frac{1}{g^{2gm}} \cdot \frac{\pi}{(g^{2m-1})^2} \cdot \left(\frac{\sqrt{\pi}}{g^m}\right)^2 = \frac{\pi}{m! g^{4m-1}} \end{aligned}$$

$$B\left(m, m\right) \cdot B\left[m + \frac{1}{g}, m + \frac{1}{g}\right] = \frac{\pi m^{-1}}{g^{4m-1}}$$

Hence proved